Do different deformations lead to the same spectrum?

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Abstract. *The procedure of $*$ quantization introduces the notions of mathematical equivalence and of $*$ spectrum. We prove that mathematical equivalence, as a change of ordering for quantum operators to which it is related, does not preserve $*$ spectrum unless it reduces to an automorphism of the $*$ product. Suggestions about the «correct» choice of $*$ products are made.*

0. INTRODUCTION

The notion of mathematical equivalence for $*$ products [2] is the usual notion of equivalence for deformations of algebras as defined originally by M. Gerstenhaber [5]. In the particular context of the Moyal $*$ product on $\mathbb{R}^{2n}$, equivalence has been related to the Weyl correspondence. More precisely, it was observed

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(see e.g. [8]) that the introduction of a weight factor in the definition of the Weyl correspondence (which in some cases corresponds to a change of ordering of the quantum operators) induces a change in the $*$ product: this change is a mathematical equivalence.

In the last years, many explicit examples of $*$ products have been built on curved phase spaces and for some of these examples a Weyl correspondence has been exhibited [2, 4, 6]. In particular on the phase space which is associated to the bound states of the hydrogen atom two different $*$ products were explicitly constructed [3]. It turns out that these $*$ products are mathematically equivalent; nevertheless they lead to different spectra in the following sense. If T is the mathematical equivalence [2] between two $*$ products $*$ and $*'$ (i.e. $T(u * v) = Tu *' Tv$, for all functions u, v on M), then the $*$ spectrum [2] of an observable H $\left(\text{i.e. the support of the Fourier transform of } \exp * \frac{t}{i\hbar} H\right)$ is the same as the $*'$ spectrum of TH; but the $*$ spectrum of H will in general be different from the $*'$ spectrum of the *same* H.

This phenomenon seems interesting for two reasons. The first is that maybe the choice of a «good» $*$ product – as the choice of quantization made by physics – can be related to geometry. The second, which is not unrelated, is to try and decide when two different $*$ products on a manifold give the same spectrum for sufficiently many observables.

Our aim in this paper is to bring some light on the second of the above mentionned questions. Though the definition of the $*$ spectrum [2] given above is perfectly satisfactory, the spectral theory for $*$ product algebras of observables is not yet as developed as the corresponding operatorial theory; moreover the $*$ analogue of projective geometry on a Hilbert space, which we shall use in the following, has not yet received an autonomous treatment. These are theories worth being studied, with potential interesting geometrical implications; for the present, we shall assume the existence of a Weyl transform and translate the question in terms of operators.

In §1 we recall the relevant definitions and we translate in operator language the isospectral problem for $*$ products. In §2 we solve the operator problem for a particular class of operators. In §3 we show how to deduce from the operator solution the fact that two isospectral, equivalent $*$ products are equal. In §4 we indicate some further possible developments.

1. Let (M, F) be a paracompact, connected symplectic manifold of class C^∞ and let $N = \mathbb{C}^\infty(M, \mathbb{R})$. Denote by $E(N, \nu)$ the space of formal power series in a parameter $\nu (\in \mathbb{C})$ with coefficients in N.

DEFINITION 1. [2] A $*$ *product* on (M, F) is a bilinear map

$$N \times N \to E(N; \nu) : (u, v) \to u * v = \sum_{r=0}^{\infty} \nu^r C_r(u, v)$$

such that

(i) $C_0(u, v) = u \cdot v$

(ii) $C_1(u, v) = \{u, v\} =$ Poisson bracket of u and v associated to the symplectic structure F.

(iii) $C_r(u, v) = (-1)^r C_r(v, u)$

(iv) $C_r(u, a) = 0, \forall a \in \mathbb{R}$

(v) $(u * v) * w = u * (v * w)$ (one extends the map $N \times N \to E(N; \nu)$ to a map $E(N; \nu) \times E(N; \nu) \to E(N; \nu)$).

DEFINITION 2. *A Weyl correspondence* W is a one-parameter family of linear maps $W^{(k)}$ $(k \in \mathbb{R})$ defined on a common algebra N' of observables onto a space of not necessarily bounded operators in a separable Hilbert space $\mathcal{H}$. A Weyl correspondence W is said to be *associated to a* $*$ *product* on M for a certain algebra of observables $\tilde{N}$, if

(i) $\tilde{N} \subset N'$

(ii) for any $f, g \in \tilde{N}$, and any $k \in \mathbb{R}$:

$$W^{(k)}(f) \circ W^{(k)}(g) = W^{(k)}(f \underset{k}{*} g)$$

where $\circ$ denotes the composition of operators, where $W^{(k)}$ is extended to complex valued functions by linearity and where:

$$f \underset{k}{*} g = (f * g)\big|_{\nu = ik} \qquad (k \in \mathbb{R})$$

$\tilde{N}$ will generally be a subalgebra of N, possibly augmented by distributions on M (such as δ) with the convention that distributions are $*$ composed with regular enough functions.

Example. When $M = \mathbb{R}^{2n}$ the usual Weyl correspondence is associated to the Moyal $*$ product; for physical applications $k = \hbar/2$. Notice also that the image of real valued functions are hermitian operators.

Assume that two $*$ products, denoted $*_1$ and $*_2$, are defined on M and assume that to each of them is associated a Weyl transform: W_1 to $*_1$ for the algebra $\tilde{N}_1$, W_2 to $*_2$ for the algebra $\tilde{N}_2$; then for $i = 1, 2$:

$$W_i(u *_i v) = W_i(u) \circ W_i(v) \qquad \forall u, v \in \tilde{N}_i.$$

Assume that W_2^{-1} is defined as a map on the range of W_1 and assume that

$W_2^{-1}W_1(\tilde{N}_1) \subset \tilde{N}_2$. Then

$$\begin{aligned} W_2^{-1}W_1(u *_1 v) &= W_2^{-1}(W_1(u) \circ W_1(v)) \\ &= W_2^{-1}(W_2(W_2^{-1}W_1(u)) \circ W_2(W_2^{-1}W_1(v)) \\ &= W_2^{-1}W_1(u) *_2 W_2^{-1}W_1(v). \end{aligned} \tag{i}$$

DEFINITION 3. [2] Two products $*_1$ and $*_2$ are said to be *mathematically equivalent* when there exists a linear map $T = \mathrm{Id} + \sum_{r \geqslant 1} \nu^r T_r$ from $N \to E(N; \nu)$ such that:

$$T(u *_1 v) = Tu *_2 Tv.$$

Remark. When the «cochains» C_r defining the $*$ products are bidifferential operators, the maps T_r are necessarily differential operators [7]. This can be easily seen by the following sequence of arguments (for the relevant definitions see [2])

(a) If the Hochshild coboundary of a 1-cochain is a differentiable 2-cochain, then the 1-cochain is local, hence locally differentiable by Peetre's theorem; an induction argument shows that in each point the order of the cochain is bounded by the order of the 2-cochain which is its coboundary; hence the 1-cochain is differentiable.

(b) An induction procedure on the equivalence series, using result (a) and the observation that

$$\begin{aligned} T &= (I + \lambda T_1 + \ldots + \lambda^p T_p) + \ldots = \\ &= (I + \lambda T_1 + \ldots + \lambda^{p-1} T_{p-1})(I + \lambda^p T_p) + O(\lambda^{p+1}) \end{aligned}$$

gives the result.

It what follows we shall restrict ourselves to mathematically equivalent $*$ products. It is common knowledge that on manifolds whose second de Rham cohomology group does not vanish there exist non equivalent $*$ products. An example of such a situation is given on the $2n$ dimensional torus $T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$. A star product $(*_1)$ is induced from Moyal on $\mathbb{R}^{2n}$; another one $(*_2)$ is given by an odd polynomial change of parameter such as $\nu \to \nu + \nu^3$ in the expression of $*_1$. One checks by direct computation that these two $*$ products are inequivalent because the symplectic 2-form on T^{2n} is not exact.

Relation (i) shows that two $*$ products with associated Weyl transforms W_1 and W_2, such that $W_2^{-1} \circ W_1$ is well defined in the sense given above are equivalent; the example of T^{2n} proves that it is not always the case; hence Weyl transforms should be handled with care.

Let $*_1$ and $*_2$ be two equivalent $*$ products on (M, F); let T be a map $N \to E(N; \nu)$ giving the equivalence. Let W_1 be a Weyl correspondence associated to $*_1$ for $\widetilde{N}_1$ and let A be a linear map defined on a space of selfadjoint operators on $\mathcal{H}$ such that:

$$W_1 T(u) = A W_1(u) \qquad u \in \widetilde{N}_1 .$$

The problem is to find conditions on $*_1$ and $*_2$ such that any function u has the same $*$ spectrum for $*_1$ and $*_2$. The spectrum of u as defined in [2] does coincide with the spectrum of the operator $W(u)$ when it is selfadjoint and W is bijective.

Finding conditions on $*_1$ and $*_2$ is clearly equivalent to finding conditions on T such that u and $T^{-1}u$ give the same spectrum for $*_1$, for any $u \in \widetilde{N}_1$. In view of our assumptions this is equivalent to finding conditions on the linear map A, such that $W_1(u)$ and $AW_1(u)$ have the same spectrum.

2. We characterize in propositions 1 and 2 below any map A of the set E of selfadjoint operators in a separable Hilbert space $\mathcal{H}$, into E, such that

(i) A is linear the sense that

$$\begin{array}{lll} \text{if } Q, R, Q+R \in E, & \text{then} & A(Q+R) = AQ + AR \\ \text{if } Q \in E, \lambda \in \mathbb{R}, & \text{then} & A(\lambda Q) = \lambda(AQ) \end{array}$$

(ii) A preserves the spectrum of every element of E, including the point spectrum with its multiplicities.

Such a map A will be called *isospectral.*

Remark. The norm of a bounded selfadjoint operator being its spectral radius, an isospectral map is automatically norm preserving.

Let us recall that an operator P in $\mathcal{H}$ is a projection operator – or projector – if and only if $P \in E$ and its spectrum consists of 0 and 1 only; furthermore its range is a finite dimensional vector space of dimension f if and only if the eigenvalue 1 has multiplicity f.

If P and P' are projection operators then $P + P' = P''$ is a projection operator if and only if the ranges F and F' of P and P' are orthogonal; the range F'' of P'' is then given by $F'' = F \oplus F'$.

From these remarks one deduces that an isospectral map A stabilizes the set of all projection operators; it also stabilizes the set of all projection operators of any given finite rank or of any given finite corank.

If P and P' are projectors in $\mathcal{H}$ whose range F and F' are orthogonal subspaces of $\mathcal{H}$, their images AP and AP' are projectors on orthogonal subspaces of $\mathcal{H}$: indeed $A(P + P') = AP + AP'$ is a projector.

If the range F of a projector P is included in the range F' of a projector P', then the range $\hat{F}$ of the projector AP is included in the range $\hat{F}'$ of the projector AP'; indeed there exists a closed subspace $\widetilde{F} = F^{\perp} \cap F'$ such that $F \oplus \widetilde{F} = F'$ and $P' = P + \widetilde{P}$, where $\widetilde{P}$ is the orthogonal projection on $\widetilde{F}$.

Let $\Pi : \mathcal{H} \setminus \{0\} \to \mathcal{H} \setminus \{0\}/_{\mathbb{C}^*} = P\mathcal{H}$ be the canonical projection on projective space and let $P^{(1)}$ be the set of projectors in $\mathcal{H}$ of rank 1. If $0 \neq x \in \mathcal{H}$, let P_x be the orthogonal projection on the 1-dimensional subspace generated by x; more generally if M is a subset of $\mathcal{H}$, P_M will denote the orthogonal projection on the closed linear subspaces generated by M and denoted $>M<$. The map

$$\alpha : P^{(1)} \to P\mathcal{H} : P_x \to \Pi(x)$$

is well defined and clearly bijective; hence A induces a map $\hat{A} : P\mathcal{H} \to P\mathcal{H}$. We shall prove that the map $\hat{A}$ lifts to a map $\widetilde{A} : \mathcal{H} \to \mathcal{H}$, which turns out to be either unitary or the composition of a unitary map and a conjugation.

LEMMA 1. a) *If L_1 and L_2 are two orthogonal points of $P\mathcal{H}$ (i.e. correspond to two orthogonal 1-dimensional subspaces in $\mathcal{H}$, also denoted L_1, L_2) their images $\hat{A}L_1$ and $\hat{A}L_2$ are two orthogonal points of $P\mathcal{H}$.*

b) *If L_1 and L_2 are two orthogonal points of $P\mathcal{H}$ and L is a point which belongs to the line determined by L_1 and L_2 then $\hat{A}L$ is a point which belongs to the line determined by $\hat{A}L_1$ and $\hat{A}L_2$.*

c) *$\hat{A}$ is injective and maps lines into lines.*

Proof. Part (a) follows from the above remarks as well as part (b); indeed $P_{L_1} + P_{L_2}$ is a projector P_M $(M = L_1 \oplus L_2)$; thus $L \subset M$ and $\hat{A}L \subset \hat{A}L_1 \oplus \hat{A}L_2$.

Injectivity follows from the fact that if $L_1 \neq L_2$ belong to $P\mathcal{H}$, then $P_{L_1} + P_{L_2}$ is a selfadjoint operator of rank 2; hence $A(P_{L_1} + P_{L_2}) = P_{\hat{A}L_1} + P_{\hat{A}L_2}$ is also of rank 2; thus $\hat{A}L_1$ and $\hat{A}L_2$ are linearly independent 1-dimensional subspaces.

For part (c) observe that any line in $P\mathcal{H}$ contains a pair of orthogonal points and apply (b).

To construct $\widetilde{A} : \mathcal{H} \to \mathcal{H}$ which is a lift of $\hat{A}$ (i.e. $AP_x = P_{\hat{A}\Pi(x)} = P_{\widetilde{A}x}$ for any $x \in \mathcal{H} \setminus \{0\}$), we shall essentially forllow Artin's proof of the fundamental theorem of projective geometry [1].

Let $\{e_i; i \in \mathbb{N}\}$ be an orthonormal basis of $\mathcal{H}$. Define e_1' by 2 conditions: $AP_{e_1} = P_{e'_1}$, $\langle e_1', e_1' \rangle = 1$. These 2 conditions define e_1' up to a phase and we choose any of the allowed e_1''s. For any $j > 1$ define e_j' to be the unique vector such that:

$$AP_{e_j} = P_{e_j'}$$
$$AP_{(e_1 + e_j)} = P_{e_1' + e_j'}.$$

This is possible in virtue of Lemma 1. The selfadjoint operators

$$\begin{cases} P_{e_1+e_j} - P_{e_1} \\ P_{e_1'+e_j'} - P_{e_1'} \end{cases}$$

must have the same spectrum. The non zero eigenvalues must respectively satisfy:

$$\nu^2 - \frac{1}{2} = 0$$

$$\nu^2 - \frac{\langle e_j', e_j'\rangle}{1 + \langle e_j', e_j'\rangle} = 0$$

and thus $\langle e_j', e_j'\rangle = 1$. Define then, for any $i \in \mathbb{N}$, $\widetilde{A}e_i = e_i'$.

For any $k \in \mathbb{C}$, define k_j $(j > 1)$ by:

$$AP_{e_1+ke_j} = P_{e_1'+k_je_j'}.$$

Clearly the map $k \to k_j$ is injective and the image of 0 is 0 and that one of 1 is 1. Now if $j \neq \ell$ and $k \neq 0$:

$$> e_j - e_\ell < \; = \; > k(e_j - e_\ell) < \; \subset \; > e_j, e_\ell <$$

and also

$$> k(e_j - e_\ell) < \; \subset \; > e_1 + ke_j, e_1 + ke_\ell <.$$

Denoting again by $\Pi : \mathcal{H} \setminus \{0\} \to P\mathcal{H}$ we have:

$$\begin{aligned} \hat{A}\Pi(e_j - e_\ell) &\subset \hat{A}(\Pi > e_1 + ke_j, e_1 + ke_\ell <) \cap \hat{A}(\Pi > e_j, e_\ell <) \\ &\subset \Pi > e_1' + k_j e_j', e_1' + k_\ell e_\ell' < \cap\, \Pi > e_j', e_\ell' < \\ &= \Pi > k_j e_j' - k_\ell e_\ell' < \\ &= \Pi > e_j' - e_\ell' < \quad \text{(taking the above identity for } k = 1) \end{aligned}$$

Hence we have $k_j = k_\ell \underset{\text{def}}{=} k'$. The map $\mathbb{C} \to \mathbb{C} : k \to k'$ is thus defined independently of j, and $\widetilde{A}(e_1 + ke_j) = e_1' + k'e_j'$.

We now extend the map $\widetilde{A}$ defined on the basis of $\mathcal{H}$, to finite linear combinations of elements of the basis by induction. Observe first that if L is a point of $P\mathcal{H}$ which belongs to the $r-1$-space $L_1 + \ldots + L_r$, then $\hat{A}L$ is a point which belongs to the $r-1$-space $\hat{A}L_1 + \ldots + \hat{A}L_r$. This is true for $r = 1, 2$, assume it is true for any $r' < r$. Then L belongs to a line $P + L_r$ where P belongs to $L_1 + \ldots + L_{r-1}$. Hence $\hat{A}L$ belongs to the line $\hat{A}P + \hat{A}L_r$ and $\hat{A}P$ belongs to $\hat{A}L_1 + \ldots + \hat{A}L_{r-1}$ by induction.

Define $\widetilde{A}(e_1 + k_2 e_2 + \ldots + k_r e_r) = e_1' + k_2' e_2' + \ldots + k_r' e_r'$; to check coherence (i.e. that $AP_x = P_{\hat{A}\Pi(x)} = P_{\widetilde{A}x}$, for such an x) one observes that:

$$> e_1 + k_2 e_2 + \ldots + k_r e_r < \subset > e_1 + k_2 e_2 + \ldots + k_{r-1} e_{r-1}, e_r <$$

$$> e_1 + k_2 e_2 + \ldots + k_r e_r < \subset > e_1 + k_r e_r, \ldots, e_{r-1} <.$$

Hence:

a $$\hat{A}\Pi > e_1 + k_2 e_2 + \ldots + k_r e_r < \subset \Pi(> e_1' + k_2' e_2' + \ldots + k_{r-1}' e_{r-1}', e_r' <)$$
$$\cap\, \Pi(> e_1' + k_r' e_r', \ldots, e_{r-1}' <)$$
$$= \Pi > e_1' + k_2' e_2' + \ldots + k_{r-1}' e_{r-1}' + k_r' e_r' <.$$

Similarly define $\widetilde{A}(k_2 e_2 + \ldots + k_r e_r) = k_2' e_2' + \ldots + k_r' e_r'$; to check coherence one observes that:

$$\hat{A}\Pi > k_2 e_2 + \ldots + k_r e_r < \subset \Pi > e_2', \ldots, e_r' < \cap\, \Pi > e_1' + k_2' e_2' + \ldots + k_r' e_r', e_1' <$$
$$= \Pi > k_2' e_2' + \ldots + k_r' e_r' <.$$

We now prove that the map $\varphi : \mathbb{C} \to \mathbb{C} : k \to k'$ associated to this construction is a field homomorphism which preserves the norm.

Indeed if $k, \ell \in \mathbb{C}$:

$$\hat{A}\Pi > e_1 + (k + \ell) e_2 + e_3 < = \Pi > e_1' + (k + \ell)' e_2' + e_3' <$$
$$\subset \Pi > e_1' + k' e_2', \ell' e_2' + e_3' <.$$

Thus $(k + \ell)' = k' + \ell'$. Similarly

$$\hat{A}\Pi > e_1 + (k\ell) e_2 + k e_3 < = \Pi > e_1' + (k\ell)' e_2' + k' e_3' <$$
$$\subset \Pi > e_1', \ell' e_2' + e_3' <.$$

Thus $(k\ell)' = k'\ell'$.

The 2 selfadjoint operators $P_{e_1} - P_{e_1 + k e_2}$, $P_{e_1'} - P_{e_1' + k' e_2'}$ have the same specturm: the non zero eigenvalues satisfy respectively:

$$\nu^2 - \frac{k\bar{k}}{1 + k\bar{k}} = 0$$

$$\nu^2 - \frac{k'\bar{k}'}{1 + k'\bar{k}'} = 0.$$

Hence $|k| = |k'|$ and $k' = k e^{i\varphi(k)}$. On the other hand

$$|1 + ke^{i\varphi(k)}|^2 = 1 + ke^{i\varphi(k)} + \overline{k}e^{-i\varphi(k)} + k\overline{k}$$
$$= 1 + k + \overline{k} + k\overline{k}$$

and thus either $k' = k$ or $k' = \overline{k}$. To summarize let us denote by $\mathfrak{I}$ the vector space of all finite linear combinations of the elements of a given basis $\{e_i; i \in \mathbb{N}\}$ of $\mathcal{H}$; we have

LEMMA 2. *There exists a map* $\widetilde{A} : \mathfrak{I} \to \mathcal{H}$ *with the following properties:*

(i) *If* $x = \sum_{i \leqslant n} \lambda_i e_i$, *either* $\widetilde{A}x = \sum_{i \leqslant n} \lambda_i e_i'$, *or* $\widetilde{A}x = \sum_{i \leqslant n} \overline{\lambda}_i e_i'$ *where* $\{e_i', i \in \mathbb{N}\}$ *is an orthonormal set of* $\mathcal{H}$.

(ii) *If* $x \in \mathfrak{I}$, $\Pi \circ \widetilde{A}(x) = \hat{A} \circ \Pi(x)$

(iii) *If* $x \in \mathfrak{I}$, $\|Ax\|^2 = \|x\|^2$

(iv) *If* $x, y \in \mathfrak{I}$, *then either* $\langle \widetilde{A}x, \widetilde{A}y \rangle = \langle x, y \rangle$, *or* $\langle \widetilde{A}x, \widetilde{A}y \rangle = \langle y, x \rangle$.

Observation. In the proof of lemma 2 we have assumed implicitly that $\dim \mathcal{H} \geqslant 3$; also we clearly define $\widetilde{A}\lambda_1 e_1$ to be either $\lambda_1 e_1'$ or $\overline{\lambda}_1 e_1'$.

We now extend $\widetilde{A}$ to arbitrary elements of $\mathcal{H}$ by defining:

$$\widetilde{A}x = \widetilde{A}\left(\sum_{i=1}^{\infty} \lambda_i e_i\right) = \begin{cases} \sum_{i=1}^{\infty} \lambda_i e_i' \\ & , \quad \sum_{i=1}^{\infty} |\lambda_i|^2 < \infty \\ \sum_{i=1}^{\infty} \overline{\lambda}_i e_i' \end{cases}$$

the choice being of course the same as for the finite dimensional linear combinations. To prove coherence of this definition, let us denote $x = x_N + u_N$, where $x_N = \sum_{i=1}^{N} x_i$. We then have:

$$\|AP_x - P_{\widetilde{A}x}\| \leqslant \|AP_x - AP_{x_N}\| + \|P_{\widetilde{A}x_N} - P_{\widetilde{A}x}\|$$
$$= \|P_x - P_{x_N}\| + \|P_{\widetilde{A}x_N} - P_{\widetilde{A}x}\|$$

(because A is linear and norm preserving)

$$\leqslant \frac{\|u_N\|}{\|x\|} + \frac{\|u_N\|}{\|x\|}$$

which proves the coherence.

The map $\widetilde{A} : \mathcal{H} \to \mathcal{H}$ is injective, linear or semilinear, preserves the norm and is an isometry onto the subspace $\widetilde{A}\mathcal{H}$ or an antiisometry onto this subspace. This implies in particular that the range $\widetilde{A}\mathcal{H}$ is a closed subspace. If this subspace is strictly contained in $\mathcal{H}$ there exists a vector $x \neq 0$ which belongs to $(\widetilde{A}\mathcal{H})^{\perp}$.

The operator $Q = \sum_{i=1}^{\infty} \frac{1}{i} P_{e_i}$ is compact and does not admit 0 as eigenvalue; the operator $AQ = \sum_{i=1}^{\infty} \frac{1}{i} P_{e'_i}$ as A is continuous in norm; it would then admit 0 as an eigenvalue which contradicts the definition of A. Hence $\widetilde{A}$ is surjective.

PROPOSITION 1. *Let A be an isospectral map. There exists a bijective map $\widetilde{A} : \mathcal{H} \to \mathcal{H}$ which is either unitary or antiunitary, such that, for any projector P in $\mathcal{H}$, the range of which is the closed subspace, F, the operator AP is the orthogonal projection on the subspace $\widetilde{A}F$.*

Proof. We have shown above the existence of a map $\widetilde{A} : \mathcal{H} \to \mathcal{H}$ which is injective and surjective and which is either linear and isometric (i.e. unitary) or antilinear and antiisometric (i.e. antiunitary), and such that

$$AP_x = P_{\widetilde{A}x} \qquad \forall x \neq 0 \in \mathcal{H}.$$

Let F be any closed subspace of $\mathcal{H}$; let $\{e_i; i = 1 \ldots N \text{ or } \infty\}$ a complete orthonormal set in F and let $\{g_j; j = 1 \ldots M \text{ or } \infty\}$ a complete orthonormal set in $F^{\perp}$. We have shown in the beginning of §2 that AP_F is a projector, say on F'; that $\widetilde{A}e_i$ is contained in F' for all i, and that $\widetilde{A}g_j$ is contained in $F'^{\perp}$ for all j. As $\widetilde{A}$ sends an orthonormal complete set in $\mathcal{H}$ on an orthonormal complete set in $\mathcal{H}$, $F' = \widetilde{A}F$ and the result follows.

PROPOSITION 2. *Let $\widetilde{A}$ be an isospectral map; then there exists a bijective unitary or antiunitary map $\widetilde{A} : \mathcal{H} \to \mathcal{H}$ such that:*

$$AQ = \widetilde{A}Q\widetilde{A}^{-1}$$

for all selfadjoint operators Q.

Proof. Proposition 1 gives the above result for any projector; the general result follows from the spectral decomposition of a selfadjoint operator and the fact that A is norm preserving hence continuous for the uniform topology.

3. We translate below the result of proposition 2 in terms of $*$ products.

Assume A is an isospectral map and K_a $(a = 1, 2)$ are two bounded selfadjoint operators, the boundedness assumption can be replaced by suitable restrictions on the domains of K_a. Then

(3.1) $$A(i[K_1, K_2]) = \pm i[AK_1, AK_2]$$

depending if $\widetilde{A}$ is linear (+) or antilinear (−).

Let (M, F) be a symplectic manifold and let $*$ denote a $*$ product on (M, F), with an associated Weyl transform W on $\widetilde{N}$. We assume that $W^{(k)}(\widetilde{N})$ is a space $\widetilde{E}$ of selfadjoint operators in a Hilbert space $\mathcal{H}$, and that $\widetilde{E}$ contains the projectors in $\mathcal{H}$; furthermore we assume that $W^{(k)}$ is injective on $\widetilde{N}$, for any k in $\mathbb{R}$. One has:

(3.2) $$W^{(k)}(u *_k v) = W^{(k)}(u) \circ W^{(k)}(v).$$

In what follows we shall only consider such $*$ products.

The $*_\nu$ spectrum of a function u is the support of the Fourier transform of $\exp * \dfrac{t}{2\nu} u$; it is equal to the spectrum of the operator $W^{(k)}(u)$ with $\nu = ik$.

LEMMA 3. *Let T^r $(r \in \mathbb{N})$ be a linear map from $\widetilde{N}$ to $\widetilde{N}$. The series $T = \mathrm{Id} + \sum_{r=1}^{\infty} \nu^{2r} T^r$ is such that u and Tu have the same $*_\nu$ spectrum for all $u \in \widetilde{N}$ and for all $\nu \in i\mathbb{R}$ if and only if T_ν is an automorphism or an antiautomorphism of the deformed braked associated to $*$; i.e. if and only if:*

(3.3) $$\pm T_\nu(u * v - v * u) = T_\nu u * T_\nu v - T_\nu v * T_\nu u.$$

Proof. Let A_k be the map defined by:

(3.4) $$W^{(k)} T_{\nu=ik}(u) = A_k W^{(k)} u \qquad (\forall u \in \widetilde{N}).$$

The map A_k is defined on $\widetilde{E}$; its values are selfadjoint operators. It is linear and preserves the spectrum in the sense of §2. Thus by proposition 2, there exists a bijective linear or semilinear map $\widetilde{A}_k : \mathcal{H} \to \mathcal{H}$ such that:

(3.5) $$A_k W^{(k)}(u) = \widetilde{A}_k W^{(k)}(u) \widetilde{A}_k^{-1} \qquad (\forall u \in \widetilde{N}).$$

From relations (3.1) and (3.2):

$$\begin{aligned} A_k(i[W^{(k)}(u), W^{(k)}(v)]) &= \pm i[A_k W^{(k)}(u), A_k W^{(k)}(v)] \\ &= \pm i[W^{(k)} T_{\nu=ik}(u), W^{(k)} T_{\nu=ik}(v)] \\ &= W^{(k)}(\pm i[T_{\nu=ik} u *_k T_{\nu=ik} v - T_{\nu=ik} v *_k T_{\nu=ik} u]) \\ &= A_k W^{(k)}[i(u *_k v - v *_k u)] \\ &= W^{(k)} i[T_{\nu=ik}(u *_k v - v *_k u)]. \end{aligned}$$

Hence the result (3.3) by injectivity of $W^{(k)}$.

LEMMA 4. [7]. *If $*_1$ and $*_2$ are two $*$ products on (M, F) which are differentiable (i.e. the cochains C_r which define the $*$ products are bidifferential operators) and such that:*

$$u *_1 v - v *_1 u = u *_2 v - v *_2 u \qquad (\forall u, v \in \tilde{N})$$

then these two $*$ products are equal.

Proof. (by induction). Assume

$$C_r(u, v) = C'_r(u, v) \qquad \forall r < 2k.$$

It is true for $r = 0, 1$ and for all odd orders. Then

$$C_{2k}(u, v) = C'_{2k}(u, v) + \tilde{\delta} E(u, v)$$

where $\tilde{\delta}$ is the Hochshild coboundary. As the $*$ products coincide at order $2k + 1$ we have:

$$\tilde{\delta}E(\{u, v\}, w) + \{\tilde{\delta}E(u, v), w\} + \{\tilde{\delta}E(v, w), u\} - \tilde{\delta}E(u, \{v, w\}) = 0.$$

Assume that E contains a term of maximal order $r \geqslant 2$.

$$E^{i_1 \dots i_r} \partial^{(r)}_{i_1 \dots i_r}$$

where $E^{i_1 \dots i_r}$ is completely symmetric. Then if we look at terms of order $(r, 1, 1)$ in (u, v, w) they appear in:

$$\Lambda^{k\ell} E^{i_1 \dots i_r} [\partial^{(r)}_{i_1 \dots i_r} (\partial_k u \cdot w)\, \partial_\ell v + \\ + \partial^{(r+1)}_{k i_1 \dots i_r} (uv)\, \partial_\ell w - \partial^r_{i_1 \dots i_r} u\, \partial_k v\, \partial_\ell w]$$

Here $\Lambda^{k\ell}$ denote the components in a local chart of $-F^{-1}$. The vanishing of the term in $\partial_{a_1 \dots a_r} u\ \partial_s v\ \partial_t w$ gives:

$$\Lambda^{a_i s} E^{t a_1 \dots \hat{a}_i \dots a_r} + \Lambda^{a_i t} E^{s a_1 \dots \hat{a}_i \dots a_r} = 0$$

which implies immediately

$$E^{t a_2 \dots a_r} = 0.$$

Hence E is of order < 2 and then $\tilde{\delta} E = 0$ and two $*$ products coincide.

PROPOSITION 3. *If two equivalent $*$ products on a symplectic manifold are isospectral (i.e. yield the same $*_\nu$ spectrum for all observable and for all $\nu = ik$ ($k \in \mathbb{R}$)) then these two $*$ products coincide.*

Proof. If $*_1$ and $*_2$ are equivalent, this equivalence can be defined [7] by a series $T = \mathrm{Id} + \sum_{r=1}^{\infty} \nu^{2r} T^r$. The $*_1$ spectrum of u is equal to the $*_2$ spectrum of Tu. As in lemma 3, T must be such that u and Tu have the same $*_2$ spectrum. Then by (3.3)

$$u *_2 v - v *_2 u = \pm (u *_1 v - v *_1 u).$$

The minus sign is excluded by the fact that the term of order 1 in any $*$ product is the Poisson bracket. The conclusion then follows from lemma 4.

4. We have shown that two equivalent, but distinct, $*$ products lead to different spectra for some observables. We want to conclude by stressing the three following points.

a. Let f be an invertible observable (i.e. there exists an observable f^{-1} such that $f * f^{-1} = f^{-1} * f = 1$). The map $u \to f * u * f^{-1}$ is clearly $*$ spectrum preserving; but this map is an *automorphism* of the $*$ product and thus does not lead to a distinct $*$ product.

b. In the definition of quantization of a classical system, an ambiguity in the ordering of operators is inherent. A change of ordering leads to different spectra for the observables. We want to stress the strong analogy between the notion of mathematical equivalence and generalized change of ordering. In the case of $\mathbb{R}^{2n}$ and the classical Weyl transform, a change of ordering is obtained by introducing a weight factor in the Weyl formula; this corresponds to an equivalence given by a series of differential operators with constant coefficients. A generalized notion of ordering can be obtained by considering weight factors which are given by a series of differential operators.

c. It seems to us, that the choice of a «good» $*$ product in a given equivalence class as the choice of a «good ordering» must be related to geometry and in particular to covariance. In the case of $\mathbb{R}^{2n}$, the Moyal $*$ product which leads to the correct physical results is the one which has maximal covariance.

d. In most physical applications one has to deal with a preferred algebra of observables, which corresponds to a certain covariance group. It woul certainly be interesting to see if some of the results proven above can be generalized in this situation and in particular to the algebra generated by the Hamiltonian itself.

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